

Multisoliton complexes on a background

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We obtain solutions of M coupled nonlinear Schrödinger equations that describe multisoliton complexes (MCs) on a background. We present explicit multiparameter families of solutions and numerical simulations, demonstrating specific features of MCs and their collisions. It is shown, in particular, that a MC on a background can have a complicated intensity profile due to a nonlinear superposition of pairs of bright and dark single solitons.

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I. INTRODUCTION

Multisoliton complexes (MCs) [1] are objects that can appear in various physical problems. In simple terms, a multisoliton complex is a self-localized state which is a nonlinear superposition of several fundamental solitons. By fundamental soliton (see, e.g., Ref. [2]) we mean the single soliton (i.e., the lowest nonlinear mode) of the system under investigation. The number of solitons in a complex depends on the nature of physical system as well as on the initial conditions. Mathematically speaking, we will talk about N fundamental solitons all having the same speed (e.g., parallel trajectories), and moving (or resting) as a single complex. The value of N can vary from $N=1$ (one fundamental soliton) and can take large values up to infinity. For a single nonlinear Schrödinger equation (NLSE) the systematic construction of N soliton solution with the same velocity has been done by Satsuma and Yajima [3]. These solutions are also known as higher-order solitons [2]. However, they are not stationary and their shape evolves during propagation. The reason is that the interaction between the fundamental solitons in this case is coherent, i.e., phase dependent. It has been demonstrated (see [4] and references therein) that stationary MC can, nevertheless, be constructed for N coupled NLSEs. In the latter case, the coherent interactions between different solitons in MC can be eliminated and the interference phenomena are suppressed.

There are various physical systems that can be described by a coupled set of nonlinear Schrödinger-type equations. The case of two coupled NLSEs have been studied in detail and various terms for solitons and MCs have been used in the existing literature. In optics, the simplest example is a “vector soliton” in a birefringent fiber, which consists of two polarization components of the pulse that are bound together by the nonlinearity [5,6]. A similar case is a soliton and its “shadow,” when the two components have independent phases and the weaker polarization component is locked in the potential created by the stronger component [7]. Solitons in nonlinear periodic structures, known as gap solitons [8], can be also described by the set of two coupled NLSEs. A parametric interaction between the two waves at different frequencies can result in their coupling and the formation of a soliton with two components [9], which is another example of a MC. A two component Bose-Einstein condensate in a

trapped ultracold gas [10] is one more example of a MC in physics. More complicated objects are solitons in multicore fiber devices [11] and incoherent solitons [4,12–14], where the number of soliton components can be more than two. In the latter case, the number of components can go up to infinity [15]. In many cases multisoliton complexes appear in conservative systems that can be Hamiltonian. However, generalization to nonconservative systems is also possible [16].

From a theoretical point of view, the most attractive case is the integrable model of a coupled set of M NLSEs, which admits exact solutions and, moreover, can be solved for arbitrary initial conditions. Quite a few interesting properties of this model have already been studied in a number of publications. These include unusual asymmetric profiles of these solitons, specific collision properties of MCs [4], etc. To make a physical picture of a MC more clear, we can use a simple analogy: one fundamental soliton has the properties of a single particle, but multisoliton complexes have nontrivial internal structure like multiparticle objects such as an exciton or an atom.

In this paper we present a solution for M coupled NLSEs that describes multisoliton complexes on a background in media with either focusing or defocusing Kerr-type nonlinearity. We demonstrate that these solutions are nonlinear superpositions of fundamental nonlinear modes: inseparable pairs of bright and dark solitons. The pairs play the role of fundamental solitons in this particular problem. As follows from our analysis, the transverse profile of the MC on the background can be quite complicated and is described by many parameters, including the amplitude of the background, the amplitudes of the fundamental solitons, and the relative distances between them. Our exact results include also collisions between MCs on a background. We show that the reshaping of the MCs after collisions are characterized by the relative shifts of the pairs of bright and dark solitons. These distinctive features of incoherent solitons are illustrated by numerical examples. Our solutions can be useful, for example, in the theory of dark incoherent solitons [17].

The paper is organized as following. We present the mathematical model in Sec. II, and, for the sake of completeness, we include in Sec. III the relevant results from Ref. [1] for bright MCs, i.e., solutions with zero boundary conditions. Then, in Sec. IV we discuss the general properties of station-

ary MCs existing on a background. In Sec. V we introduce a sophisticated rotation transformation which allows us to extend the technique developed for bright solitons, and obtain exact solutions for MCs with a nonzero background. Finally, numerical examples are given in Sec. VI.

II. MATHEMATICAL MODEL

An incoherent soliton is a highly nontrivial example of a MC. We will frame our analysis to this problem as it is the most representative one. The evolution of incoherent solitons in a medium with a Kerr-like nonlinearity can be described by the following set of NLSEs [4,18,19]:

$$i \frac{\partial \psi_m}{\partial z} + \frac{1}{2} \frac{\partial^2 \psi_m}{\partial x^2} + \delta n(I) \psi_m = 0, \quad (1)$$

where ψ_m is the field in the m th component of the beam ($m=1,2,3,\dots,M$), z is the coordinate along the direction of propagation, x is the transverse coordinate, and

$$\delta n(I) = \sum_{m=1}^M \alpha_m |\psi_m|^2 \quad (2)$$

is the change of refractive index, induced by all the M components, where the α_m are weighting coefficients. From the physical model it follows that all the α_m should have the same sign, $s = \text{sgn}(\alpha_m)$. In a self-focusing nonlinear medium, $s = +1$, while in defocusing material, $s = -1$.

For further analysis it is convenient to rewrite Eqs. (1) for a set of normalized functions $u_m(x, z) = \sqrt{|\alpha_m|} \psi_m(x, z)$,

$$i \frac{\partial u_m}{\partial z} + \frac{1}{2} \frac{\partial^2 u_m}{\partial x^2} + s u_m \sum_{j=1}^M |u_j|^2 = 0. \quad (3)$$

When $M=1$, the set of equations reduces to a single nonlinear Schrödinger equation which is known to be completely integrable [20]. The inverse scattering technique (IST) for the set of two equations ($M=2$) has been developed as well [21]. Moreover, it has been shown that the coupled set of M equations ($M>2$) is also completely integrable [22,23].

The results of the IST tell us that the solution for this integrable model consists of a number of solitons plus radiation. The former is defined by the discrete part and the latter by the continuous part of the IST spectrum. We are interested in pure soliton solutions when the radiation is absent. In general, the number of solitons might exceed the number of equations M . Stationary solutions are possible only when all the fundamental solitons have the same velocity and each of them is polarized in (i.e., belongs to) a different component, so that their total number is equal to M . This is the case that we consider below. The collisions of MCs will also be studied, but under the same assumption of orthogonal polarizations of all the fundamental solitons composing the colliding MCs.

First, consider the case when the background is zero. Then the bright solitons must be supported by a self-focusing nonlinearity ($s = +1$). Corresponding MCs were studied earlier, and exact solutions for $M \leq 4$ have been presented in explicit form [24]. Then, a solution describing propagation and collisions of bright MCs when the number of solitons

and the number of equations are arbitrary independent parameters has been derived [1]. It has been shown that coherent and incoherent interactions between the fundamental solitons result in some unique features of the MC. As we mentioned earlier, in this paper we will concentrate on the case where each component contains just one fundamental soliton, so that the internal interactions in the complex are incoherent.

III. BRIGHT MC WITH ZERO BOUNDARY CONDITIONS

Bright MC solutions of Eq. (3) composed of orthogonally polarized fundamental solutions can be found from the set of linear equations [25,26]:

$$\sum_{m=1}^M \frac{e_j e_m^* u_m}{k_j + k_m^*} + \frac{1}{2r_j} u_j = -e_j, \quad (4)$$

where $e_j = \chi_j \exp(k_j \bar{x}_j + ik_j^2 \bar{z}_j / 2)$, and the shifted coordinates are $\bar{x}_j = x - x_j$ and $\bar{z}_j = z - z_j$. This result comes from considering the MCs as generalized reflectionless potentials [27]. Each fundamental soliton is characterized by the shifts along the axes x_j and z_j and by the wave number $k_j = r_j + i\mu_j$. The amplitude of the fundamental soliton is related to r_j , while its motion in the transverse direction is determined by the imaginary part, $\mu_j = \tan \theta_j$, where θ_j is the angle of propagation relative to the z axis. The coefficients χ_j can be arbitrary, but, in order to obtain the solutions in a symmetric form, we have to choose them in a special way [1,24]:

$$\chi_j = \prod_{m \neq j} \sqrt{b_{jm}}, \quad (5)$$

where $b_{jm} = (k_j + k_m^*) / (k_j - k_m)$, and the square root value is taken on the branch with the argument in the limits $(-\pi/2, \pi/2)$.

The solution of Eqs. (4) and (5) describing multisoliton complexes can be obtained in explicit form [1]

$$u_j = \frac{e^{i\gamma_j}}{U} \sum_{\{1, \dots, j-1, j+1, \dots, M\} \rightarrow L} C_L^j F_L^j(x, z), \quad (6)$$

$$U = \sum_{\{1, \dots, M\} \rightarrow L} C_L F_L(x, z),$$

where

$$C_L = T_{mb}, \quad C_L^j = 2r_j \chi_j T_{mb}, \quad (7)$$

$$F_L = \cosh(S_b), \quad F_L^j = \cosh(S_b^j).$$

Here L refers to sets of indices (L_1, L_2) , and the summation goes over all possible permutations of soliton numbers between the two sets. Then, the variables for each realization of L are found to be

$$T_{mb} = \prod_{j \in L_1; m \in L_2} c_{jm}, \quad S_b = \sum_{m \in L_1} \beta_m - \sum_{m \in L_2} \beta_m,$$

$$S_b^j = S_b + i \sum_{m \in L_1} \varphi_{jm} - i \sum_{m \in L_2} \varphi_{jm}, \quad (8)$$

where $\beta_j + i\gamma_j = k_j \bar{x}_j + i k_j^2 \bar{z}_j / 2$ (with β_j and γ_j real), $c_{jm} = |b_{jm}|$, and $\varphi_{jm} = \arg(1/b_{jm})/2$. Note that only β_j and γ_j depend on the coordinates (x, z) . All the other coefficients are expressed in terms of the wave numbers k_j and constant shifts in positions (x_j, z_j) of the M fundamental solitons. As the solution has translational symmetry along the x axis, one of the shifts can be fixed, so that the number of independent parameters controlling the multisoliton complex is $2M - 1$.

IV. STATIONARY MC SOLUTIONS ON A BACKGROUND

It has already been demonstrated in [28] that MCs can exist on a constant background. However, until now only symmetric, sech-type solutions with two free parameters have been found [28]. Clearly, the variety of possible MC on a background is much wider. Indeed, as we demonstrated above, each fundamental soliton in the complex must be controlled by two independent parameters. Moreover, there is an additional characteristic in this problem, namely the amplitude of the background. The method that we use in this paper allows us to present a full multiparameter family of solutions.

One of the interesting features of MCs on a background, which follows from this analysis, is that they can be decomposed into a number of elementary objects. Each of them has two parts, (i) bright, or intensity peak in one component, and (ii) dark, or a hole in the background. We note that from the point of view of the IST such an object is still a single fundamental soliton, although physically it is a coupled set of bright and dark counterparts. When these objects are located on top of each other, they comprise a nonlinear superposition which has a complicated transverse profile. Curiously enough, these simple objects, as well as their nonlinear superpositions, exist for both signs of the nonlinearity, $s = \pm 1$.

A useful complementary view of a multisoliton complex is to consider it as a self-induced optical waveguide [4]. The existence of the background does not change the concept. The outside part of the waveguide might have any constant refractive index. For simplicity, let us first consider stationary multicomponent evolution, which can be observed if all the fundamental soliton velocities are zero, i.e., $\mu_j = 0$. Then, the component fields are $u_m(x, z) = U_m(x) \exp(i\lambda_m z)$, where the real amplitudes U_m are determined from the set of ordinary differential equations ($1 \leq m \leq M$):

$$\frac{1}{2} \frac{\partial^2 U_m}{\partial x^2} - \lambda_m U_m + U_m V(x) = 0, \quad (9)$$

which follows from the original system (3). This can be viewed as a linear problem of eigenfunctions and eigenvalues. However, the potential $V(x) = s \sum_{j=1}^M |u_j|^2$ must be self-consistent. In general, the latter condition can be achieved in numerical modeling with the help of iterative schemes [30]. However, for the problem at hand we will be able to derive exact analytical solutions due to the integrability of the original Eqs. (3).

Let us analyze the properties of the system (9). In our problem, all the field changes correspond to solitons, and thus are localized. Then, the boundary conditions are such that the potential acquires a constant value at infinity, namely $V(x) \rightarrow V_0$ at $x \rightarrow \pm\infty$. The case $V_0 = 0$ corresponds to bright MCs, with the solutions given by Eqs. (6)–(8). On the other hand, $V_0 \neq 0$ means that the field in at least one of the components does not vanish at infinity. In this case, mutual trapping of bright and dark solitons may occur.

The self-consistent potential is the same for all components, and thus we can apply the Sturm-Liouville theorem to Eq. (9). In particular, the two important properties are the following: (i) bright fundamental solitons correspond to localized eigenstates, with $\lambda_m > -V_0$, and (ii) only one nonoscillatory *dark mode* is possible, with its eigenvalue located exactly at the boundary between the discrete and continuous spectra, i.e., $\lambda_m = -V_0$. Quite remarkably, the general conclusions also hold for nonstationary MCs dynamics involving soliton collisions, with the only requirement that there is no more than one bright soliton in any component. This is demonstrated below.

V. SOLUTION FOR MC ON A BACKGROUND

As has been demonstrated in the preceding section, a MC on a nonoscillatory background should contain a dark mode with an eigenvalue lying at the top of the effective potential well. As a first step in constructing this solution, let us identify such mode in a bright MC with zero boundary conditions, $V_0 = 0$. Then, we take the expressions for a complex of bright solitons (6), and consider a limit

$$k_M = r_M \rightarrow +0, \quad (10)$$

so that the corresponding component profile will approach that of a dark mode, provided that the limiting transformation is done properly. It is now convenient to return to the system of linear equations (4). We note that, from the explicit solution (6)–(8), it follows that the amplitude of the pseudodark mode is vanishingly small, $u_M \rightarrow 0$, which is consistent with the fact that a bright MC does not contain dark solitons. As a result, the last linear equation in Eq. (4) becomes decoupled, and after enforcing the limit (10) we have

$$v_M = -1 - J. \quad (11)$$

At this stage of the calculations, we introduce the new functions

$$v_m(x, z) = u_m(x, z) / k_m^*, \quad (12)$$

so that $J = \sum_{m=1}^{M-1} e_m^* v_m$. This sum depends only on the amplitudes of the other components, which in turn are found from an independent system of linear equations, resulting from Eq. (4):

$$\sum_{m=1}^{M-1} \frac{e_j e_m^* v_m k_m^*}{k_j + k_m^*} + \frac{1}{2r_j} v_j k_j^* = -e_j, \quad (13)$$

where $1 \leq j \leq M - 1$.

After finding the expression for the dark component profile, we need to develop a special technique to construct solutions for MCs with a nonzero background. As mentioned earlier, the most nontrivial problem is how to satisfy the self-consistency relation for the effective potential. To address this issue, we note that the self-induced waveguide (or potential well) depends only on the mode intensities. Thus, important information can be obtained by looking at the normalized intensity profile of the pseudodark mode, which can be found using Eqs. (11): $|v_M|^2 = 1 + J + J^* + |J|^2$. To calculate this value, we first multiply Eq. (13) by v_j^* , then sum over j and, finally, add a complex conjugate expression. Then, we arrive to the following equality:

$$\sum_{m=1}^M |v_m|^2 = 1 \quad \text{when } k_M \rightarrow 0. \quad (14)$$

This remarkable result demonstrates an intrinsic relation between the intensities of bright and dark solitons. Moreover, this relation opens up an opportunity to introduce a *rotation transformation* in functional space and construct solutions for MC on a background, with the dark component having a nonzero amplitude. Specifically, we can change the bright soliton intensities in such a way that the new potential is $V_{\text{new}} = V_0 + V_{\text{old}}$, where a free parameter $|V_0|$ is the background intensity. Then, by adjusting the propagation constants, the *self-consistency condition for the effective potential can be preserved*, and this is the principal feature of the introduced rotation transformation. Moreover, it is also possible to obtain a solution for a *self-defocusing medium*. Finally, the resulting MC solution on background is

$$u_m(x, z) = v_m(x, z) \sqrt{s|k_m|^2 + sV_0} \exp(iV_0 z). \quad (15)$$

We recall that, according to Eq. (10), the pseudodark mode wave number is $k_M = 0$. Some limitations for the background amplitude can be immediately obtained from Eq. (15): (i) $V_0 \geq 0$ in a self-focusing medium ($s = +1$), and (ii) $V_0 \leq -\max_m |k_m|^2$ in a defocusing material ($s = -1$).

As follows from the method of constructing the solution, the MC consists of a nonlinear superposition of $M - 1$ solitons and a plane wave in the M th component. It is also clear that the plane wave must change its profile at the places where the bright solitons are located. If the number of components, M , is 2, then it can be seen from Eqs. (14) and (15) that the solution is a nonlinear superposition of the dark and the bright soliton parts. This superposition exist in pairs. In a self-focusing medium a ‘‘hole’’ on a background is compensated for by the higher amplitude of the corresponding bright soliton. On the other hand, pure bright solitons cannot exist in a defocusing material, and they must be supported by the waveguides created by the corresponding dark counterparts. The envelopes in a separated pair have the profiles

$$\begin{aligned} u_m &= e^{i\gamma_m + iV_0 z} r_m \sqrt{s + sV_0|k_m|^{-2}} \operatorname{sech}(\beta_m), \\ u_M &= e^{iV_0 z} \sqrt{sV_0} [r_m \tanh(\beta_m) + i\mu_m] / k_m^*. \end{aligned} \quad (16)$$

which are a known pair of dark and bright single solitons for the coupled set of Manakov equations, found previously by

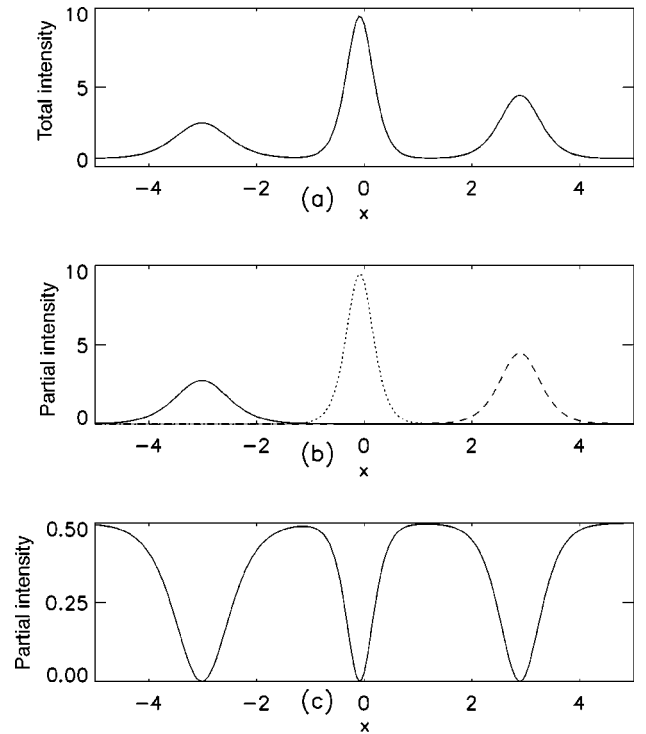


FIG. 1. Three stationary solitons on a background in a self-focusing medium ($s = +1$): (a) intensity profile; (b) bright soliton profiles shown with solid, dashed, and dotted lines; (c) background intensity. The parameters for this simulation are the following: $k_1 = 1.5$, $k_2 = 3$, $k_3 = 2$, $x_1 = -2$, $x_2 = 0$, $x_3 = 2$.

Inoue [29]. When the number of components M is more than 2, these pairs are combined into more complicated superpositions.

The bright solitons belonging to different components interact incoherently due to the nature of the coupling term in the model Eqs. (1). Compared to the situation considered in the present paper, coherent coupling between the bright solitons in one component is quite a different process, resulting, for example, in spatial beating [1]. Another interesting fact to note is that the ‘‘holes’’ in the background are black, not gray, solitons, when their velocities are zero. Indeed, as the black component is the highest mode of the self-induced wave guide, there are as many zeros in the profile as the number of lower-order ‘‘bright’’ modes, which is $M - 1$. Then, each zero is created by a single black soliton.

VI. NUMERICAL EXAMPLES

A simple example of a multisoliton complex on a background, when the relative distances between the fundamental solitons are large, is shown in Fig. 1. In this case, the pairs of dark and bright solitons are asymptotically separated in the transverse direction and hardly interact with each other. Their profiles can be found using Eqs. (16). Note that the bright solitons belong to different components but the dark solitons are all in the same mode.

In general, the structure of the solution is more complicated. A case where all the fundamental solitons are located near to each other is shown in Fig. 2. The actual intensity profile is determined by the soliton eigenvalues, or wave numbers, and the relative distances between the pairs of dark

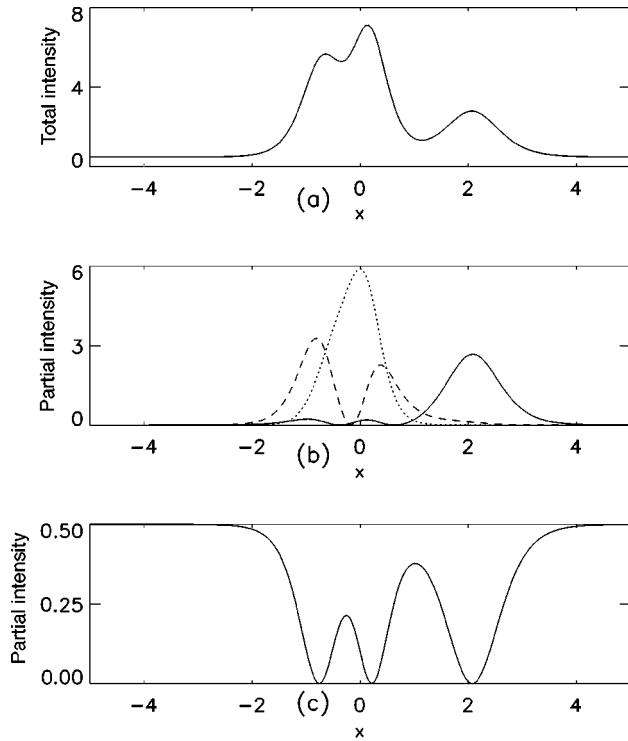


FIG. 2. Multisoliton complex composed of the pairs of bright and dark solitons with the same eigenvalues as shown in Fig. 1, but located closer together: $x_1=0.05$, $x_2=0$, $x_3=0.2$.

and bright solitons. If the relative distances are nonzero, MCs are asymmetric, as demonstrated in Figs. 1 and 2, but they can be symmetric if there are no shifts between the fundamental solitons. This is similar to the case of MCs with zero background [4].

An example of a symmetric MC on a background is shown in Fig. 3. In this case, all relative shifts are equal to zero. Further simplification appears when the eigenvalues are chosen in a special way, namely, they are all multiples of the consecutive integers. Then the intensity profile of the MC becomes a sech^2 function on a background and the solutions coincide with those found in [28].

VII. COLLISIONS OF MC ON A BACKGROUND

The numerical examples presented above are for the case of stationary MCs with zero velocity. When the velocity is not zero, different MCs can collide, and these phenomena are also described by our explicit solution. A numerical example demonstrating the collision of two MCs on a background is shown in Fig. 4. The presence of the background does not change the common rule: velocities of the MC after the collision are the same as before the collision. Another principal feature is that the multisoliton complexes change their shape after collision and these changes are similar to those which occur without the background [4]. In the particular case shown in Fig. 4, the multisoliton complex is almost decomposed into its basic nonlinear constituents, i.e., pairs of bright and dark solitons. The acquired shifts along the x axis of each of the fundamental solitons are found to be

$$\delta x_j = \frac{1}{r_j} \sum_m f_{jm} \ln(c_{jm}). \quad (17)$$

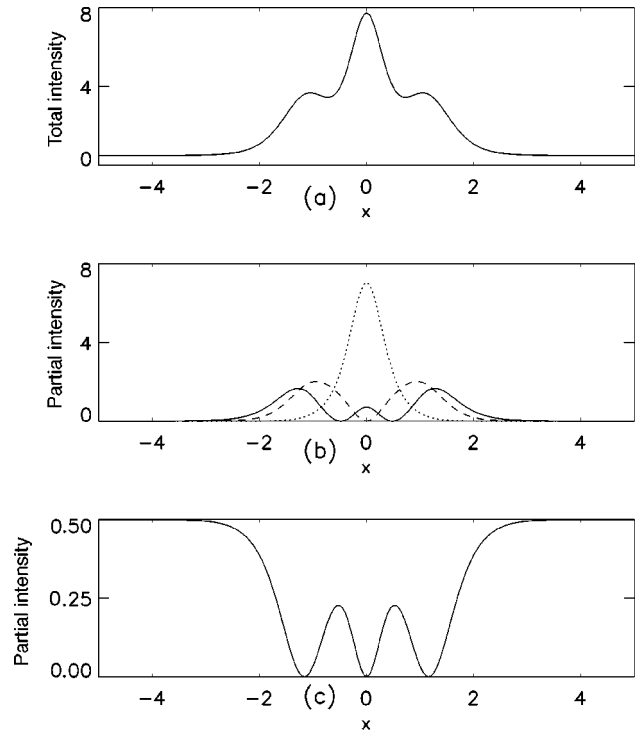


FIG. 3. Multisoliton complex composed of the same pairs of bright and dark solitons as shown in Fig. 1, but with no relative shifts, i.e., $x_j=0$.

Here the summation involves the fundamental solitons which feature in the collisions. When the colliding soliton number m comes from the right (i.e., has a larger x coordinate before the impact), we put $f_{jm} = +1$ while we set $f_{jm} = -1$ when from the left. Note that the presented expression is exact for a separated fundamental soliton, because interactions between the solitons in a MC may result in additional shifts. However, if after a collision all the shifts calculated accord-

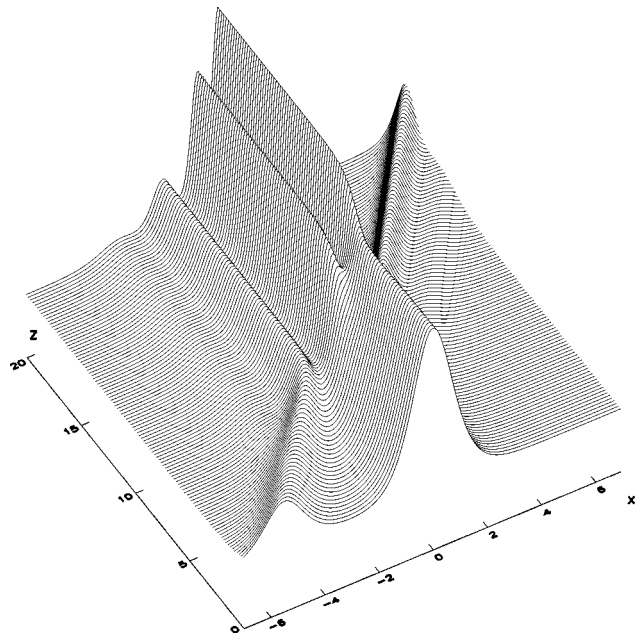


FIG. 4. Collision of two and four component multisoliton complexes existing on a background in a self-focusing medium.

ing to Eq. (17) are the same, than no corrections are needed, and the MC profile remains unchanged. This fact has been confirmed by numerical simulations for bright MCs [4], but is also valid when the background is present.

The integrable model that we have used to describe the multisoliton complexes is unique in several aspects. Its advantage is that it allows us to write the solutions analytically and that these solutions describe them completely. On the other hand, the soliton pairs in a multisoliton complex do not have any binding energy and in this sense they stay in the complex due to properly chosen initial conditions. If the nonlinearity is different from the Kerr-type, then the binding energy will keep all soliton components in the complex (or repel them), depending on the sign of the binding energy. The case of saturable nonlinearity requires more study, but a zero background case has been already investigated in Ref. [30]. It has been found that a MC becomes a nonstationary oscillating beam after collisions.

Special case of only dark solitons

Pure dark solitons, supported by defocusing nonlinearity ($s = -1$), were extensively investigated earlier [31]. Our solution (15) can be reduced to describe such a case, giving a link to previous studies. To do this, we choose the wave numbers according to [20], i.e. in such a way that $|V_0| = |k_m|^2$, where $1 \leq m < M$. Then, the amplitudes of all the bright solitons reduce to zero, and the resulting expression gives a multidark soliton solution. An example of three dark soliton collisions is presented in Fig. 5.

VIII. CONCLUSIONS

In conclusion, we have obtained a solution for M coupled nonlinear Schrödinger equations that describes stationary multisoliton complexes in media with a Kerr-type nonlinearity. A particular case is multisoliton complexes on a background, which can exist both in self-focusing and defocusing media. The solutions are formed as nonlinear superpositions of pairs of bright and dark solitons located close to each other. The transverse profile of a MC can be quite compli-

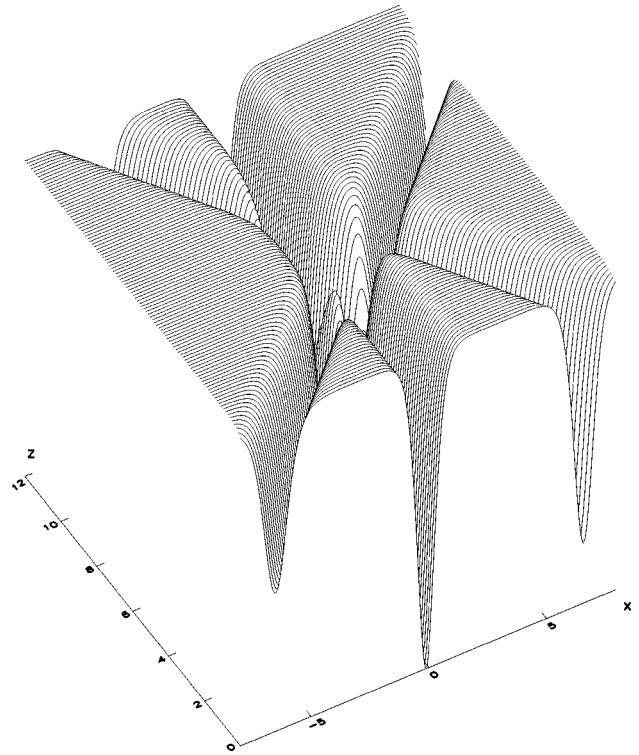


FIG. 5. Collision of three purely dark solitons in a defocusing medium.

cated and is described by many parameters, including the amplitudes of the solitons, relative distances between them, and the background intensity. Our exact results also describe collisions between MCs on a background and their subsequent reshaping, which is characterized by the relative shifts of the pairs of bright and dark solitons. These distinctive features of incoherent solitons are illustrated by numerical examples.

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